

HOMEWORK

Section 4.4 - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 23, 27, 30, 31, 32, 35, 37, 42, 44, 55, 57, 58

SECTION 4.4 - MATRICES: BASIC OPERATIONS

Addition and Subtraction. First, let's define what it means for two matrices to be equal.

Definition 1 (Equal). *Two matrices are equal if they are the same size and the corresponding elements in each matrix are equal.*

For example, the equality

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

is true if and only if

$$\begin{array}{l} a = u \quad b = v \\ c = w \quad d = x \\ e = y \quad f = z \end{array} .$$

In order to add or subtract matrices **they must be the same size.**

- When adding matrices, we just add the corresponding elements.
- When subtracting matrices, we just subtract the corresponding elements.

Example 1. *Find the indicated operations*

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Solution.

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3+(-2) & 2+3 \\ -1+1 & -1+(-1) \\ 0+2 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & -2 \\ 2 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3-2 & 2-(-2) \\ 5-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -4 \end{bmatrix}$$

(c) *These matrices are not the same size and so cannot be added.***Example 2.** *Find the indicated operations*

(a)

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + [-2 \quad 3 \quad -2]$$

Scalar Multiplication. If k is a number and M is a matrix, we can form the scalar product kM by just multiplying every element of M by k .**Example 3.** *Find*

$$-2 \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

Solution.

$$-2 \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(3) & -2(-1) & -2(0) \\ -2(-2) & -2(1) & -2(3) \\ -2(0) & -2(-1) & -2(-2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

Example 4. *Find*

$$5 \begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & -3 \\ 3 & 3 \end{bmatrix}$$

Matrix Multiplication. In order to define matrix multiplication, it is easier to first define the product of a row matrix with a column matrix.**Definition 2.** *Suppose we have a $1 \times n$ row matrix A and an $n \times 1$ column matrix B where*

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$AB = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

It is very important that the number of columns in A matches the number of rows in B .

Example 5. Find

$$[-1 \ 0 \ 3 \ 2] \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$$

Solution.

$$[-1 \ 0 \ 3 \ 2] \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = (-1)(2) + (0)(3) + (3)(4) + (2)(-1) = -2 + 0 + 12 - 2 = 8$$

Example 6. Find

$$[2 \ -1 \ 1] \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Definition 3 (Matrix Multiplication). Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix. Let R_i denote the matrix formed by the i^{th} row of A and let C_j denote the matrix formed by the j^{th} column of B . Then the ij^{th} element of the matrix product AB is R_iC_j .

Remark 1. It is very important that the number of columns of A matches the number of rows of B , otherwise the products R_iC_j would not be able to be defined. That is, if A is an $m \times n$ matrix and B is an $p \times q$ matrix, the product AB is defined if and only if $n = p$.

Example 7. Let $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $D =$

$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. Find the following products, if possible.

- (a) AB
- (b) BA
- (c) CD
- (d) DC
- (e) CB
- (f) D^2

Solution.

(a) Since A is 2×4 and B is 3×2 , the product AB is not defined.

(b)

$$\begin{aligned}
 BA &= \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 \\ R_3C_1 & R_3C_2 & R_3C_3 & R_3C_4 \end{bmatrix} \\
 &= \begin{bmatrix} [-1 \ 1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [-1 \ 1] \begin{bmatrix} 0 \\ 2 \end{bmatrix} & [-1 \ 1] \begin{bmatrix} 3 \\ 2 \end{bmatrix} & [-1 \ 1] \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ [2 \ 3] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [2 \ 3] \begin{bmatrix} 0 \\ 2 \end{bmatrix} & [2 \ 3] \begin{bmatrix} 3 \\ 2 \end{bmatrix} & [2 \ 3] \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ [1 \ 0] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [1 \ 0] \begin{bmatrix} 0 \\ 2 \end{bmatrix} & [1 \ 0] \begin{bmatrix} 3 \\ 2 \end{bmatrix} & [1 \ 0] \begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} (-1)(-1) + (1)(1) & (-1)(0) + (1)(2) & (-1)(3) + (1)(2) & (-1)(-2) + (1)(0) \\ (2)(-1) + (3)(1) & (2)(0) + (3)(2) & (2)(3) + (3)(2) & (2)(-2) + (3)(0) \\ (1)(-1) + (0)(1) & (1)(0) + (0)(2) & (1)(3) + (0)(2) & (1)(-2) + (0)(0) \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}
 \end{aligned}$$

(c)

$$\begin{aligned}
 CD &= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} (1)(-2) + (2)(1) & (1)(4) + (2)(-2) \\ (-1)(-2) + (-2)(1) & (-1)(4) + (-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

Solution.

(d) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(e) *Not defined.*

(f) $\begin{bmatrix} 8 & -16 \\ -4 & 8 \end{bmatrix}$

Remark 2. Note that parts (c) and (d) show that matrix multiplication is not commutative. That is, it is not necessarily true that $AB = BA$ for matrices A and B , even if both matrix products are defined.